# Higher-Order Approximations for Symmetrical Regular Long Wave Equation

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In this work, we extend the application of "the modified reductive perturbation method" to the symmetrical regular long wave equation and try to obtain the contribution of higher-order terms in the perturbation expansion. It is shown that the lowest-order term in the expansion is governed by the nonlinear Schrödinger equation while the second- and third-order terms are governed by the linear Schrödinger equation. By employing the hyperbolic tangent method, progressive wave type solutions are obtained for the first-, second- and third-order terms in the perturbation expansion. – PACS numbers: 02.30.Jr, 42.25.Bs, 42.81.Dp, 43.35.Kp.

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## 1. Introduction

Asymptotic and perturbation analysis have played a significant role in theoretical physics and applied mathematics. This method is to continuously deform a simple problem easy to solve into the difficult problem under study. In many cases, regular perturbation methods are not applicable, and various singular perturbation techniques must be used. Examples of widely used techniques include the homotopy perturbation method [1], the Adomian decomposition method [2], and various Lindstedt-Poincare methods [3]. Although these methods are well known, each has its own drawbacks, preventing mechanical (or algorithmic) application. In most perturbation methods it is assumed that a small parameter exists, but most nonlinear problems have no small parameter. Furthermore, equations whose highest-order derivatives are multiplied by a small parameter $\varepsilon$ often yield solutions with narrow regions of rapid variation.

So far, many new methods have been proposed to eliminate the small parameter in the equation. Among them, the reductive perturbation method [4] has been introduced by use of the stretched time and space variables to study the higher-order terms in the perturbation expansion. The negligence of higher-order dispersive effects in the classical reductive perturbation method leads to the imbalance of nonlinearity and dispersion, which results in some secular terms in the solution of the evolution equations. Recently, in [5] the "modified reductive perturbation method" is presented, which provides a more natural and effective way to find weakly dispersive ion-acoustic plasma waves and solitary waves in an elastic tube filled with fluid. By employing the hyperbolic tangent method a progressive wave type of solution can be sought and possible secularities can be removed.

The method of modified reductive perturbation, for instance, is applied for finding solutions of partial differential equations:

$$\Lambda(\phi, \phi_x, \phi_t, \phi_{xt}, \ldots) = 0. \quad (1)$$

Following [5], we introduce the stretched variables $\xi = \varepsilon(x - \lambda t)$, $\tau = \varepsilon^2 g t$, that is $\phi = \phi(x, t, \xi, \tau)$. Furthermore, we assume that an approximate solution has the form

$$\phi = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + O(\varepsilon^4). \quad (2)$$

Substituting expression (2) into (1) and equating coefficients of like powers of $\varepsilon$, we obtain equations for determining $\phi_1$, $\phi_2$, $\phi_3$. In fact, the basic idea in this method was the inclusion of higher-order dispersive effects through the introduction of the scaling parameters $g$ and $\lambda$ in a special way, to balance the higher-order nonlinearities with dispersion. Furthermore, the scaling parameters are obtained as part of the solution so as to remove the possible secularities in the solution.

In the present work, we extend the application of the modified reductive perturbation method to the symmetrical regular long wave equation to examine the contributions of higher-order terms in the perturbation expansion. It is shown that the lowest-order term in the expansion is governed by the nonlinear Schrödinger equation, whereas the second- and third-order terms are governed by the linear Schrödinger equation with inhomogeneous terms. In the long-wave limit, progressive wave type solutions are obtained for the first-, second- and third-order terms of the perturbation expansion. Besides, the scale parameters are obtained as part of the solution so as to remove the possible secularities in the solution.

## 2. Modified Reductive Perturbation Method for Symmetrical Regular Long Wave Equation

In this section we focus our attention on the investigation of the symmetrical regular long wave equation [6] given by

$$\phi_t + \alpha\psi_x + \beta(\phi^3)_x + \gamma\phi_{xxt} = 0, \quad \psi_t = \phi_x,$$

where $\alpha$, $\beta$ and $\gamma$ are constants. It describes the propagation of ion-plasma acoustic waves in space under weakly nonlinear action. [6] gives the soliton solutions of it in the case of $\alpha = -1$, $\beta = 1/3$ and $\gamma = -1$. Clearly, differentiating the first formula once with respect to time and eliminating $\psi$, the formula can be turned into

$$\phi_{tt} + \alpha\phi_{xx} + \beta(\phi^3)_{xt} + \gamma\phi_{xxtt} = 0. \tag{3}$$

First, the following stretched coordinates may be introduced:

$$\xi = \varepsilon(x - \lambda t), \quad \tau = \varepsilon^2 g t, \tag{4}$$

where $\varepsilon$ is the smallness parameter, $\lambda$ and $g$ are the scale parameters that characterize the higher-order dispersive effects. Therefore, we shall assume that $\lambda$ and $g$ depend on the smallness parameter $\varepsilon$ and may be expanded into an asymptotic series of $\varepsilon$ as follows:

$$\lambda = \lambda_0 + \sum_{n=1}^{\infty} \varepsilon^n \lambda_n, \quad g = 1 + \sum_{n=1}^{\infty} \varepsilon^n g_n. \tag{5}$$

We can for instance use

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \varepsilon^3\lambda_3,$$
$$g = 1 + \varepsilon g_1 + \varepsilon^2 g_2 + \varepsilon^3 g_3.$$

Hence the following transformations are valid:

$$\begin{aligned} \frac{\partial}{\partial x} &\to \frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial \xi}, \\ \frac{\partial}{\partial t} &\to \frac{\partial}{\partial t} - \varepsilon\lambda_0\frac{\partial}{\partial \xi} + \varepsilon^2\left(\frac{\partial}{\partial \tau} - \lambda_1\frac{\partial}{\partial \xi}\right) \\ &\quad + \sum_{n=1}^{\infty}\varepsilon^{n+2}\left(g_n\frac{\partial}{\partial \tau} - \lambda_{n+1}\frac{\partial}{\partial \xi}\right). \end{aligned} \tag{6}$$

We shall further assume that the field quantities depend on the fast variables $(x,t)$, the slow variables $(\xi,\tau)$ and the smallness parameter $\varepsilon$. The field variable $\phi$ may be expanded into an asymptotic series as

$$\phi = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + \varepsilon^4\phi_4 + \varepsilon^5\phi_5 + \cdots. \tag{7}$$

Introducing (6) and (7) into the field equation (3) and setting the coefficients of like powers of $\varepsilon$ equal to zero, a set of equations at different orders of $\varepsilon$ are obtained:

$\varepsilon^1$ order equation:

$$\alpha\frac{\partial^2\phi_1}{\partial x^2} + \gamma\frac{\partial^4\phi_1}{\partial x^2\partial t^2} + \frac{\partial^2\phi_1}{\partial t^2} = 0; \tag{8}$$

$\varepsilon^2$ order equation:

$$\begin{aligned} &\alpha\frac{\partial^2\phi_2}{\partial x^2} + \gamma\frac{\partial^4\phi_2}{\partial x^2\partial t^2} + \frac{\partial^2\phi_2}{\partial t^2} + 2\gamma\frac{\partial^4\phi_1}{\partial t^2\partial x\partial \xi} \\ &+ 2\alpha\frac{\partial^2\phi_1}{\partial x\partial \xi} - 2\lambda_0\frac{\partial^2\phi_1}{\partial t\partial \xi} - 2\gamma\lambda_0\frac{\partial^4\phi_1}{\partial t\partial x^2\partial \xi} = 0; \end{aligned} \tag{9}$$

$\varepsilon^3$ order equation:

$$\begin{aligned} &\alpha\frac{\partial^2\phi_3}{\partial x^2} + \gamma\frac{\partial^4\phi_3}{\partial x^2\partial t^2} + \frac{\partial^2\phi_3}{\partial t^2} + 2\gamma\frac{\partial^4\phi_1}{\partial t\partial \tau\partial x^2} + 2\frac{\partial^2\phi_1}{\partial t\partial \tau} \\ &+ 2\alpha\frac{\partial^2\phi_2}{\partial x\partial \xi} + 2\gamma\frac{\partial^4\phi_2}{\partial t^2\partial x\partial \xi} + 3\beta\phi_1^2\frac{\partial^2\phi_1}{\partial t\partial x} + \gamma\frac{\partial^4\phi_1}{\partial t^2\partial \xi^2} \\ &-4\gamma\lambda_0\frac{\partial^4\phi_1}{\partial t\partial x\partial \xi^2} + \lambda_0^2\frac{\partial^2\phi_1}{\partial \xi^2} + 6\beta\phi_1\frac{\partial\phi_1}{\partial t}\frac{\partial\phi_1}{\partial x} \\ &-2\gamma\lambda_0\frac{\partial^4\phi_2}{\partial t\partial x^2\partial \xi} + \gamma\lambda_0^2\frac{\partial^4\phi_1}{\partial x^2\partial \xi^2} - 2\lambda_0\frac{\partial^2\phi_2}{\partial t\xi} \\ &+ \alpha\frac{\partial^2\phi_1}{\partial \xi^2} = 0; \end{aligned} \tag{10}$$

$\varepsilon^4$ order equation:

$$\alpha\frac{\partial^2\phi_4}{\partial x^2} + \gamma\frac{\partial^4\phi_4}{\partial t^2\partial x^2} + \frac{\partial^2\phi_4}{\partial t^2} + 6\beta\lambda_0\phi_1\frac{\partial\phi_1}{\partial \xi}\frac{\partial\phi_1}{\partial x}$$

$$+ 6\beta\phi_1\frac{\partial\phi_1}{\partial t}\frac{\partial\phi_1}{\partial \xi} + 6\beta\phi_1\frac{\partial\phi_1}{\partial t}\frac{\partial\phi_2}{\partial x} + 6\beta\phi_2\frac{\partial\phi_1}{\partial t}\frac{\partial\phi_1}{\partial x}$$

$$
\begin{aligned}
&+6\beta\phi_1\frac{\partial\phi_2}{\partial t}\frac{\partial\phi_1}{\partial x}+2\frac{\partial^2\phi_2}{\partial t\partial\tau}+3\beta\phi_1^2\frac{\partial^2\phi_2}{\partial t\partial x}-2\gamma\lambda_0\frac{\partial^4\phi_3}{\partial t\partial x^2\partial\xi}\\
&+\gamma\lambda_0^2\frac{\partial^4\phi_2}{\partial x^2\partial\xi^2}-4\gamma\lambda_0\frac{\partial^4\phi_1}{\partial t\partial x\partial\xi^2}-2\gamma\lambda_0\frac{\partial^4\phi_1}{\partial\tau\partial x^2\partial\xi}\\
&-2\gamma\lambda_0\frac{\partial^4\phi_1}{\partial t\partial\xi^3}+6\beta\phi_1\phi_2\frac{\partial^2\phi_1}{\partial t\partial x}-3\beta\lambda_0\phi_1^2\frac{\partial^2\phi_1}{\partial x\partial\xi}\\
&-2\lambda_1\frac{\partial^2\phi_1}{\partial x\partial\xi}+2\gamma\lambda_0^2\frac{\partial^4\phi_1}{\partial x\partial\xi^3}+2\gamma g_1\frac{\partial^4\phi_1}{\partial t\partial\tau\partial x^2}\\
&-2\gamma\lambda_1\frac{\partial^4\phi_1}{\partial t\partial x^2\partial\xi}+3\beta\phi_1^2\frac{\partial^2\phi_1}{\partial t\partial\xi}-2\lambda_0\frac{\partial^2\phi_3}{\partial t\partial\xi}-2\lambda_0\frac{\partial^2\phi_1}{\partial\tau\partial\xi}\\
&+2g_1\frac{\partial^2\phi_1}{\partial t\partial\tau}+2\alpha\frac{\partial^2\phi_3}{\partial x\partial\xi}+\alpha\frac{\partial^2\phi_2}{\partial\xi^2}+\gamma\frac{\partial 4\phi_2}{\partial t^2\partial\xi^2}+\lambda_0^2\frac{\partial^2\phi_2}{\partial\xi^2}\\
&+2\gamma\frac{\partial^4\phi_3}{\partial t^2\partial x\partial\xi}+4\gamma\frac{\partial^4\phi_1}{\partial t\partial\tau\partial x\partial\xi}+2\gamma\frac{\partial^4\phi_2}{\partial t\partial\tau\partial x^2}=0; \quad (11)
\end{aligned}
$$

$\varepsilon^5$ order equation:

$$\alpha\frac{\partial^2\phi_5}{\partial x^2}+\gamma\frac{\partial^4\phi_5}{\partial t^2\partial x^2}+\frac{\partial^2\phi_5}{\partial t^2}+\cdots, \quad (12)$$

and so on; we omit the following expressions for simplicity.

## 3. Solution of the Field Equations

In this section we shall present a solution to the field equations given in (8) – (12). As it is well known, if the nonlinear effects are small, the system of equations that describes the physical phenomenon admits a harmonic wave solution with constant amplitude. If the amplitude of the wave is small-but-finite, the nonlinear terms cannot be neglected and the nonlinearity gives rise to the variation of amplitude both in space and time variables. Then the amplitude varies slowly over a period of oscillation. A stretching transformation allows us to decompose the system into a rapidly varying part associated with the oscillation and a slowly varying part such as the amplitude. Now, seeking a harmonic wave type of solution to (8) we have

$$\phi_1=\phi_1^{(1)}(\xi,\tau)\exp[\mathrm{i}(kx-\omega t)]+\text{c.c.}, \quad (13)$$

where $\omega$ is the angular frequency, $k$ is the wave number and c.c. stands for the complex conjugate of the corresponding expression.

In order to have a nonzero solution for $\phi_1^{(1)}(\xi,\tau)$ the following dispersion relation must be satisfied:

$$\omega^2=\frac{\alpha k^2}{\gamma k^2-1}. \quad (14)$$

Introducing (14) into (9) we have

$$
\begin{aligned}
&\alpha\frac{\partial^2\phi_2}{\partial x^2}+\gamma\frac{\partial^4\phi_2}{\partial x^2\partial t^2}+\frac{\partial^2\phi_2}{\partial t^2}\\
&-2\mathrm{i}(-\alpha k+\gamma\lambda_0\omega k^2-\lambda_0\omega+\gamma\omega^2 k)\frac{\partial\phi_1^{(1)}}{\partial\xi}\mathrm{e}^{\mathrm{i}\varphi} \quad (15)\\
&+\text{c.c.}=0,
\end{aligned}
$$

where the phase $\varphi$ is defined by $\varphi=kx-\omega t$. We shall seek a solution to this equation of the form

$$\phi_2=\sum_{l=1}^{2}\phi_2^{(l)}\exp(\mathrm{i}l\varphi)+\text{c.c.} \quad (16)$$

As is seen from (15), it is $\phi_2^{(2)}=0$. In order to have a nonzero solution for $\phi_1^{(1)}$ the following condition must be satisfied:

$$\lambda_0=-\frac{\alpha k}{(\gamma k^2-1)^2\omega}=\frac{\mathrm{d}\omega}{\mathrm{d}k} \quad \text{(group velocity)}. \quad (17)$$

Here, $\phi_2^{(1)}$ remains as another unknown function to be determined from higher-order contributions.

To obtain the solution of the $\varepsilon^3$ order equation, we introduce the solutions given in (13) and (16) into (10) and obtain

$$
\begin{aligned}
&\alpha\frac{\partial^2\phi_3}{\partial x^2}+\gamma\frac{\partial^4\phi_3}{\partial x^2\partial t^2}+\frac{\partial^2\phi_3}{\partial t^2}\\
&+\Big\{(\lambda_0^2-4\gamma\lambda_0\omega k-\gamma\lambda_0^2k^2+\alpha-\gamma\omega^2)\frac{\partial^2\phi_1^{(1)}}{\partial\xi^2}\\
&\quad+2\omega\mathrm{i}(\gamma k^2-1)\frac{\partial\phi_1^{(1)}}{\partial\tau}+3\beta\omega k|\phi_1^{(1)}|^2\phi_1^{(1)}\Big\}\mathrm{e}^{\mathrm{i}\varphi} \quad (18)\\
&+9\beta\omega k(\phi_1^{(1)})^3\mathrm{e}^{3\mathrm{i}\varphi}+\text{c.c.}=0,
\end{aligned}
$$

Equation (18) admits a harmonic solution of the form

$$\phi_3=\sum_{l=1}^{3}\phi_3^{(l)}\exp(\mathrm{i}l\varphi)+\text{c.c.} \quad (19)$$

Introducing (19) into (18) we obtain the nonlinear Schrödinger equation

$$\mathrm{i}\frac{\partial\phi_1^{(1)}}{\partial\tau}+P\frac{\partial^2\phi_1^{(1)}}{\partial\xi^2}+Q|\phi_1^{(1)}|^2\phi_1^{(1)}=0, \quad (20)$$

provided that the following relations hold true:

$$
\begin{aligned}
&\phi_3^{(2)}=0,\\
&\phi_3^{(3)}=-\frac{\beta\omega k}{-\omega^2+9\gamma\omega^2k^2-\alpha k^2}(\phi_1^{(1)})^3, \quad (21)\\
&\gamma k^2=2,
\end{aligned}
$$

where

$$P=\frac{\lambda_0^2+\alpha-\gamma\omega^2-4\gamma\lambda_0\omega k-\gamma\lambda_0^2k^2}{2\omega(-1+\gamma k^2)},\quad Q=\frac{3\beta\omega k}{2\omega(-1+\gamma k^2)}. \tag{22}$$

To obtain the solution of the $\varepsilon^4$ order equation we introduce the solutions given in (13), (16) and (19) into (11), and obtain

$$\begin{aligned}&\alpha\frac{\partial^2\phi_4}{\partial x^2}+\gamma\frac{\partial^4\phi_4}{\partial x^2\partial t^2}+\frac{\partial^2\phi_4}{\partial t^2}+\Bigg\{\left[-\mathrm{i}(Pk^2\beta+2Q\lambda_0)\frac{\partial\phi_1^{(1)*}}{\partial\xi}+2Q\omega\phi_2^{(1)*}\right](\phi_1^{(1)})^2\\&+\left[-6\beta\mathrm{i}(\lambda_0k+\omega)\phi_1^{(1)*}\frac{\partial\phi_1^{(1)}}{\partial\xi}+4Q\phi_2^{(1)}\phi_1^{(1)*}\omega\right]\phi_1^{(1)}+2\omega\left[\mathrm{i}\frac{\partial\phi_1^{(1)}}{\partial\tau}g_1-\mathrm{i}\lambda_1\frac{\partial\phi_1^{(1)}}{\partial\xi}+\mathrm{i}\frac{\partial\phi_2^{(1)}}{\partial\tau}+\gamma P\frac{\partial^2\phi_2^{(1)}}{\partial\xi^2}\right]\\&\qquad+\mathrm{i}2\gamma(\lambda_0^2k+\lambda_0\omega)\frac{\partial^3\phi_1^{(1)}}{\partial\xi^3}+2(-\lambda_0+2\gamma k\omega+\gamma\lambda_0k^2)\frac{\partial^2\phi_1^{(1)}}{\partial\tau\partial\xi}\Bigg\}\mathrm{e}^{\mathrm{i}\varphi}\\&+\left\{6\mathrm{i}(-9\gamma k^3\alpha-9\gamma\lambda_0k^2\omega+\alpha k+\lambda_0\omega)\frac{\partial\phi_3^{(3)}}{\partial\xi}+\left[-9\beta\mathrm{i}(\omega+\lambda_0k)\frac{\partial\phi_1^{(1)}}{\partial\xi}+6Pk^2\phi_2^{(1)}\right](\phi_1^{(1)})^2\right\}\mathrm{e}^{3\mathrm{i}\varphi}+\text{c.c.}=0.\end{aligned} \tag{23}$$

Again, this equation admits a solution of the form

$$\phi_4=\sum_{l=1}^{4}\phi_4^{(l)}\exp(\mathrm{i}l\varphi)+\text{c.c.} \tag{24}$$

Introducing (24) into (23) has the resulting equation

$$\begin{aligned}&\mathrm{i}\frac{\partial\phi_2^{(1)}}{\partial\tau}+P\frac{\partial^2\phi_2^{(1)}}{\partial\xi^2}+2Q|\phi_1^{(1)}|^2\phi_2^{(1)}\\&\qquad+Q(\phi_1^{(1)})^2\phi_2^{(1)*}=\Lambda_1(\phi_1^{(1)}),\end{aligned} \tag{25}$$

where $\Lambda_1(\phi_1^{(1)})$ is defined by

$$\Lambda_1(\phi_1^{(1)})=\frac{3}{k}\frac{\partial^2\phi_1^{(1)}}{\partial\tau\partial\xi}+\mathrm{i}g_1\frac{\partial\phi_1^{(1)}}{\partial\tau}-\mathrm{i}\lambda_1\frac{\partial\phi_1^{(1)}}{\partial\xi}, \tag{26}$$

and

$$\phi_4^{(2)}=\phi_4^{(4)}=0,\ \ \phi_4^{(3)}=-\frac{3k\beta}{16\omega}(\phi_1^{(1)})^2\phi_2^{(1)}. \tag{27}$$

The left-hand side of (25) is the linear Schrödinger equation in terms of $\phi_2^{(1)}$ and the right-hand side is an inhomogeneous source term in terms of $\phi_1^{(1)}$.

Finally, for the solution of the $\varepsilon^5$ order equation, we introduce the solutions given in (13), (16), (19) and (24) into (12) and obtain

$$\begin{aligned}&\alpha\frac{\partial^2\phi_5}{\partial x^2}+\gamma\frac{\partial^4\phi_5}{\partial x^2\partial t^2}+\frac{\partial^2\phi_5}{\partial t^2}+\gamma\Bigg\{3\omega^2\frac{\partial^2\phi_3^{(1)}}{\partial\xi^2}+\mathrm{i}\Big(2Qk^2|\phi_1^{(1)}|^2+k^2g_2\omega\Big)\frac{\partial\phi_1^{(1)}}{\partial\tau}+2Q\omega\phi_1^{(1)*}\frac{\partial\phi_1^{(1)}}{\partial\xi}^2\\&+\Big(-\mathrm{i}\lambda_2\omega k^2+4Q\omega\phi_1^{(1)}\frac{\partial\phi_1^{(1)*}}{\partial\xi}\Big)\frac{\partial\phi_1^{(1)}}{\partial\xi}+\Big(-Pk^3\lambda_1+2Q\omega|\phi_1^{(1)}|^2\Big)\frac{\partial^2\phi_1^{(1)}}{\partial\xi^2}+\mathrm{i}\omega\frac{\partial\phi_3^{(1)}}{\partial\tau}k^2+Pk^3\frac{\partial^2\phi_2^{(1)}}{\partial\tau\partial\xi}\\&-\mathrm{i}\lambda_1\omega\frac{\partial\phi_2^{(1)}}{\partial\xi}k^2+2Q\omega k^2|\phi_2^{(1)}|^2\phi_1^{(1)}+Qk^2(\phi_2^{(1)})^2\omega\phi_1^{(1)*}+Q\omega k^2(\phi_1^{(1)})^2\phi_3^{(1)*}+Pk^5\beta|\phi_1^{(1)}|^2\phi_3^{(1)}\\&+\mathrm{i}Qk^2(\phi_1^{(1)})^2\frac{\partial\phi_1^{(1)*}}{\partial\tau}-\frac{1}{16}Q\beta k^3|\phi_1^{(1)}|^4\phi_1^{(1)}+2\mathrm{i}\omega\frac{\partial^3\phi_1^{(1)}}{\partial\tau\partial\xi^2}+\alpha\frac{\partial^4\phi_1^{(1)}}{\partial\xi^4}-\frac{1}{2}\frac{\partial^2\phi_1^{(1)}}{\partial\tau^2}k^2+\mathrm{i}g_1\omega\frac{\partial\phi_2^{(1)}}{\partial\tau}k^2\\&+g_1Pk^3\frac{\partial^2\phi_1^{(1)}}{\partial\tau\partial\xi}+Q\omega(\phi_1^{(1)})^2\frac{\partial^2\phi_1^{(1)*}}{\partial\xi^2}\Bigg\}\mathrm{e}^{\mathrm{i}\varphi}-\frac{3\beta}{8k}\phi_1^{(1)}\Bigg\{27\mathrm{i}k^2\phi_1^{(1)}\frac{\partial\phi_1^{(1)}}{\partial\tau}+22\omega\frac{\partial\phi_1^{(1)}}{\partial\xi}^2+11\omega\phi_1^{(1)}\frac{\partial^2\phi_1^{(1)}}{\partial\xi^2}\\&-24\phi_1^{(1)}\phi_3^{(1)}Pk^4-24(\phi_2^{(1)})^2Pk^4+6Qk^2|\phi_1^{(1)}|^2(\phi_1^{(1)})^2\Bigg\}\mathrm{e}^{3\mathrm{i}\varphi}-\frac{25}{8}Q\beta k(\phi_1^{(1)})^5\mathrm{e}^{5\mathrm{i}\varphi}+\text{c.c.}=0.\end{aligned} \tag{28}$$

We shall propose a solution to (21) of the form

$$\phi_5 = \sum_{l=1}^{5} \phi_5^{(l)} \exp(\mathrm{i} l \varphi) + \text{c.c.} \tag{29}$$

In (28), for our future purposes we only need the equation proportional to $e^{i\varphi}$, which reads

$$\mathrm{i}\frac{\partial \phi_3^{(1)}}{\partial \tau} + P\frac{\partial^2 \phi_3^{(1)}}{\partial \xi^2} + 2Q|\phi_1^{(1)}|^2\phi_3^{(1)} + Q(\phi_1^{(1)})^2\phi_3^{(1)*} = \Lambda_2(\phi_1^{(1)}, \phi_2^{(1)}), \tag{30}$$

where the function $\Lambda_2(\phi_1^{(1)}, \phi_2^{(1)})$ is defined by

$$\begin{aligned}
&\Lambda_2(\phi_1^{(1)}, \phi_2^{(1)}) = \\
&-\frac{3\beta}{k}\phi_1^{(1)*}\frac{\partial \phi_1^{(1)}}{\partial \xi}^2 + \Big(\mathrm{i}\lambda_2 - \frac{6\beta}{k}\phi_1^{(1)}\frac{\partial \phi_1^{(1)*}}{\partial \xi}\Big)\frac{\partial \phi_1^{(1)}}{\partial \xi} \\
&+ \mathrm{i}\lambda_1\frac{\partial \phi_2^{(1)}}{\partial \xi} - \Big(\mathrm{i}g_2 + \frac{3\mathrm{i}k\beta}{\omega}|\phi_1^{(1)}|^2\Big)\frac{\partial \phi_1^{(1)}}{\partial \tau} + \frac{1}{2\omega}\frac{\partial^2 \phi_1^{(1)}}{\partial \tau^2} \\
&-\frac{3k\beta}{2}\phi_1^{(1)*}(\phi_2^{(1)})^2 - 3k\beta\phi_1^{(1)}|\phi_2^{(1)}|^2 \\
&-\frac{3\mathrm{i}k\beta}{2\omega}(\phi_1^{(1)})^2\frac{\partial^2 \phi_1^{(1)*}}{\partial \tau} - \mathrm{i}g_1\frac{\partial \phi_2^{(1)}}{\partial \tau} - \frac{3}{k}\frac{\partial^2 \phi_2^{(1)}}{\partial \tau \partial \xi} \\
&-\frac{\omega}{k^4}\frac{\partial^4 \phi_1^{(1)}}{\partial \xi^4} - \frac{2\mathrm{i}}{k^2}\frac{\partial^3 \phi_1^{(1)}}{\partial \tau \partial \xi^2} - \frac{3g_1}{k}\frac{\partial^2 \phi_1^{(1)}}{\partial \tau \partial \xi} + \frac{3\lambda_1}{k}\frac{\partial^2 \phi_1^{(1)}}{\partial \xi^2} \\
&+\frac{3k^2\beta^2}{32\omega}|\phi_1^{(1)}|^4\phi_1^{(1)} - \frac{3\beta}{k}|\phi_1^{(1)}|^2\frac{\partial^2 \phi_1^{(1)}}{\partial \xi^2} \\
&-\frac{3\beta}{2k}(\phi_1^{(1)})^2\frac{\partial^2 \phi_1^{(1)*}}{\partial \xi^2}.
\end{aligned} \tag{31}$$

Again, the left-hand side of (30) is a linear Schrödinger equation and the right-hand side is an inhomogeneous term in terms of $\phi_1^{(1)}$ and $\phi_2^{(1)}$.

The other equations of the asymptotic series can be determined in a similar way.

## 4. Progressive Wave Solution

In this section we shall try to find the progressive wave solution to (20), (25) and (30). For that purpose we introduce

$$\phi_1^{(1)} = \mu(\zeta)\exp[\mathrm{i}(K\xi - \Omega\tau)], \quad \zeta = \xi - v_\mathrm{p}\tau, \tag{32}$$

where $\mu(\zeta)$ is an unknown real function of its argument, $K$, $\Omega$ and the speed $v_\mathrm{p}$ of progressive waves are some constants to be determined from the solution. Introducing (32) into (20) we obtain

$$P\frac{\partial^2 \mu}{\partial \zeta^2} + (\Omega - K^2P)\mu + Q\mu^3 = 0, \tag{33}$$

provided that $v_\mathrm{p} = 2PK$. This indeed coincides with the result of the classical nonlinear Schrödinger equation under progressive wave solution (see for instance [7]). Obviously, (33) admits the following type of solitary wave solution:

$$\mu = A\mathrm{sech}\left[\left(\frac{Q}{2P}\right)^{1/2}A\zeta\right], \tag{34}$$

where $QP > 0, \Omega = PK^2 - QA^2/2$, the speed $v_\mathrm{p}$ of the progressive and the speed $v_\mathrm{e}$ of enveloping waves are, respectively, given by $v_\mathrm{p} = 2PK$, $v_\mathrm{e} = PK - QA^2/(2K)$, and

$$\mu = A\tanh\left[\left(-\frac{Q}{2P}\right)^{1/2}A\zeta\right], \tag{35}$$

where $QP < 0$, $\Omega = PK^2 - QA^2$, the speed $v_\mathrm{p}$ of the progressive and the speed $v_\mathrm{e}$ of enveloping waves are, respectively, given by $v_\mathrm{p} = 2PK$, $v_\mathrm{e} = PK - QA^2/K$. Here, in both cases $A$ stands for the amplitude of the solitary waves. In order not to repeat, we select the zero boundary travelling solitary wave solution (34) as an example in the following.

To solve (25) again we assume a solution for $\phi_2^{(1)}$ of the form

$$\phi_2^{(1)} = h(\zeta)\exp[\mathrm{i}(K\xi - \Omega\tau)], \tag{36}$$

where $h(\zeta)$ is a complex function of its argument. Introducing (32), (34) and (36) into (25) and (26) we obtain

$$\Big(2\mathrm{i}k^2\frac{\partial h}{\partial \zeta}v_\mathrm{p} - 2k^2h\Omega - 6\omega\frac{\partial^2 h}{\partial \zeta^2} - 12\mathrm{i}\omega\frac{\partial h}{\partial \zeta}K + 6\omega hK^2\Big)\sinh(\chi\zeta)\cosh(\chi\zeta)^3$$

$$+ 2\Big(-g_1Ak^2\Omega + 3Ak\chi^2v_\mathrm{p} - 3AkK\Omega - \lambda_1Ak^2K\Big)\sinh(\chi\zeta)\cosh(\chi\zeta)^2$$

$$-2\Big(\mathrm{i}g_1Ak^2\chi v_\mathrm{p}+3\mathrm{i}AkK\chi v_\mathrm{p}+3\mathrm{i}Ak\chi\Omega+\mathrm{i}\lambda_1Ak^2\chi\Big)\sinh(\chi\zeta)^2\cosh(\chi\zeta)$$
$$-3\Big(2\beta A^2hk^3+\beta A^2h^*k^3\Big)\sinh(\chi\zeta)\cosh(\chi\zeta)-12Ak\chi^2v_\mathrm{p}\sinh(\chi\zeta)=0, \quad (37)$$

where for convenience, we have defined $\chi^2=QA^2/(2P)$. The form of (37) indicates that $h(\zeta)$ is complex, i. e., $h(\zeta)=h_1(\zeta)+\mathrm{i}h_2(\zeta)$, the real and imaginary parts must satisfy the following sets of equations:

$$\begin{aligned}&2\Big(-k^2\Omega h_1-3\omega\frac{\partial^2h_1}{\partial\zeta^2}+3\omega K^2h_1)\sinh(\chi\zeta)\cosh(\chi\zeta)\Big)^3\\&+2\Big(-3AkK\Omega-g_1Ak^2\Omega-\lambda_1Ak^2K+\frac{18A\chi^2\omega K}{k}\Big)\sinh(\chi\zeta)\cosh(\chi\zeta)^2\\&-9\beta k^3A^2h_1\sinh(\chi\zeta)\cosh(\chi\zeta)-\frac{72A\chi^2\omega K}{k}\sinh(\chi\zeta)=0,\end{aligned} \quad (38)$$

$$\begin{aligned}&2\Big(-k^2\Omega h_2+3\omega K^2h_2-3\omega\frac{\partial^2h_2}{\partial\zeta^2}\Big)\cosh(\chi\zeta)^3\\&-3\beta k^3A^2h_2\cosh(\chi\zeta)+2\Big(-\lambda_1Ak^2\chi-3Ak\chi\Omega\\&-6g_1A\chi\omega K-\frac{18AK^2\chi\omega}{k}\Big)\sinh(\chi\zeta)\cosh(\chi\zeta)=0.\end{aligned} \quad (39)$$

By employing the hyperbolic tangent method a progressive wave type solution

$$h_1=-\frac{3AK}{k}\mathrm{sech}(\chi\zeta),\quad h_2=0 \quad (40)$$

is sought and possible secularities are removed by selecting the scaling parameters as

$$\lambda_1=\frac{9(\beta k^3A^2+4\omega K^2)}{4k^3},\quad g_1=-\frac{6K}{k}. \quad (41)$$

For the solution of (30) we again assume a solution for $\phi_3^{(1)}$ of the form

$$\phi_3^{(1)}=s(\zeta)\exp[\mathrm{i}(K\xi-\Omega t)], \quad (42)$$

where $s(\zeta)$ is a complex function of its argument. Introducing (42) into (30) we obtain

$$\begin{aligned}&\Big(3\omega k^2\frac{\partial^2s}{\partial\zeta^2}-\frac{3}{4}\beta A^2sk^5\Big)\cosh(\chi\zeta)^5\\&+\Big(-8v_\mathrm{p}k^2\mathrm{i}A\chi^3+18\mathrm{i}\beta A^3k^3\chi K\Big)\sinh(\chi\zeta)\cosh(\chi\zeta)\\&+c_1\cosh(\chi\zeta)^4+\Big(3\beta A^2sk^5+\frac{3}{2}\beta A^2k^5s^*\Big)\cosh(\chi\zeta)^3\\&+c_2\sinh(\chi\zeta)\cosh(\chi\zeta)^3+c_3\cosh(\chi\zeta)^2\\&+24\omega A\chi^4-\frac{3}{32\omega}\beta^2A^5k^6-18\beta A^3k^3\chi^2=0,\end{aligned} \quad (43)$$

where

$$\begin{aligned}c_1={}&\frac{9}{32\omega}\beta^2A^5k^6+\lambda_2Ak^4K+6\beta A^3k^3K^2\\&-\frac{33}{4}\beta A^3k^3\chi^2-\frac{55}{2}\omega AK^4+3\omega Ak^2g_2K^2\\&-\frac{3}{4}A^3k^5g_2\beta+\omega A\chi^4+141\omega AK^2\chi^2,\\c_2={}&\mathrm{i}\Big(v_\mathrm{p}k^4Ag_2\chi-110\omega AK^3\chi+8\omega A\chi^3K\\&+A\lambda_2k^4\chi+\frac{51}{2}\beta A^3k^3\chi K\Big),\\c_3={}&-\frac{9}{8\omega}\beta^2A^5k^6-282\omega AK^2\chi^2-20\omega A\chi^4\\&+\frac{87}{2}\beta A^3k^3K^2+30\beta A^3k^3\chi^2.\end{aligned} \quad (44)$$

The form of (43) reveals that $s(\zeta)$ must be complex, e. g., $s(\zeta)=s_1(\zeta)+\mathrm{i}s_2(\zeta)$, the real and imaginary parts must satisfy the following differential equations:

$$\begin{aligned}&-24\beta A^2k^5\omega s_1\cosh(\chi\zeta)^5+\Big(32\lambda_1\omega Ak^4K\\&+192\beta\omega A^3k^3K^2+9\beta^2A^5k^6+4512\omega^2AK^2\chi^2\\&+96Ak^2g_1\omega^2K^2-880\omega^2AK^4-264\beta\omega A^3k^3\chi^2\\&-24A^3k^5g_2\omega\beta+32\omega^2A\chi^4\Big)\cosh(\chi\zeta)^4\\&+144\beta A^2k^5\omega s_1\cosh(\chi\zeta)^3+\Big(960\beta\omega A^3k^3\chi^2\\&+1392\beta\omega A^3k^3K^2-36\beta^2A^5k^6-640\omega^2A\chi^4\\&-9024\omega^2AK^2\chi^2\Big)\cosh(\chi\zeta)^2+768\omega^2A\chi^4\\&-576\beta\omega A^3k^3\chi^2-3\beta^2A^5k^6=0,\end{aligned} \quad (45)$$

$$\Big[\Big(102\beta A^3k^3\chi\omega K+24\omega^2Ak^2g_2\chi K+4\lambda_2\omega Ak^4\chi$$

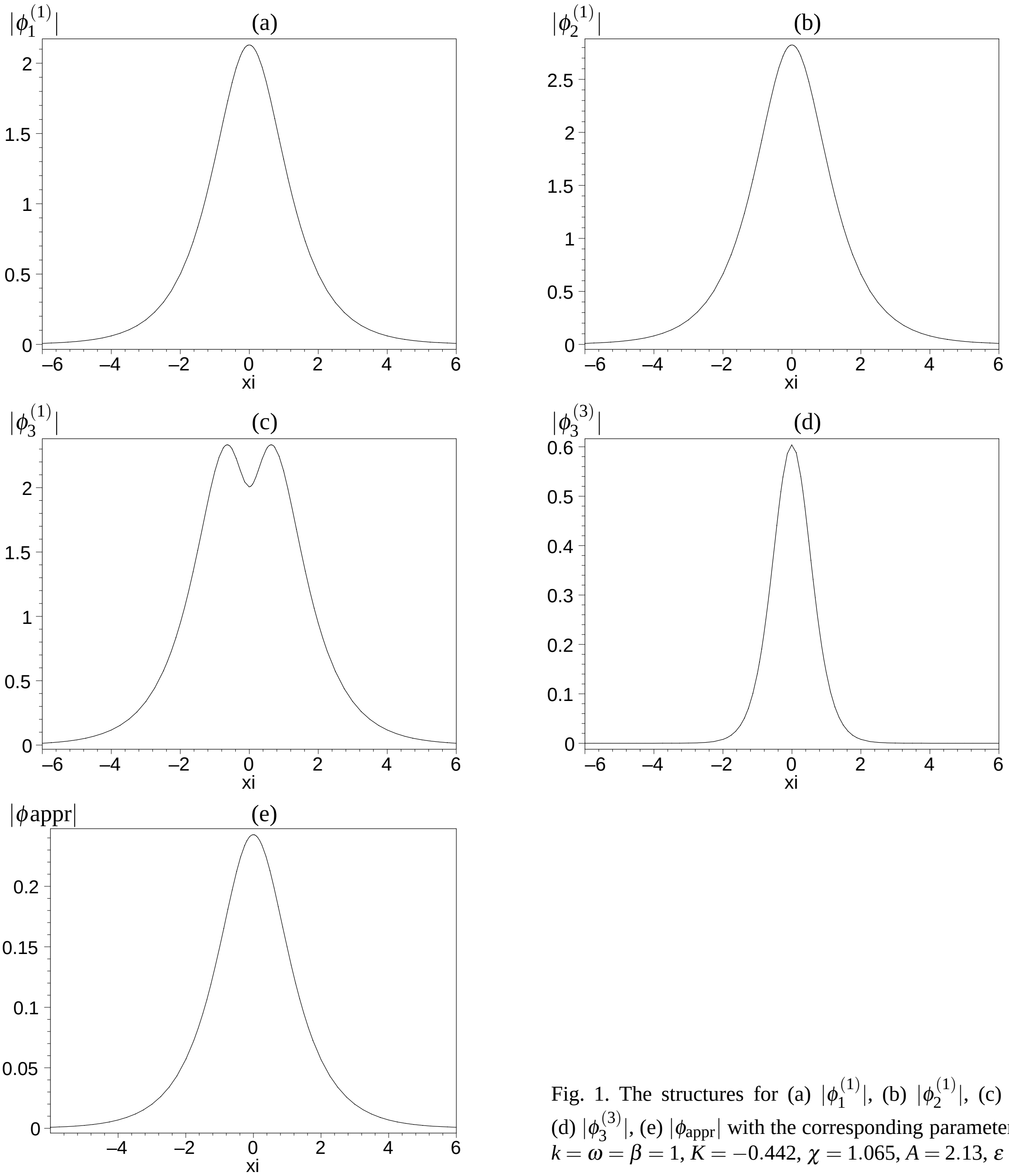


Fig. 1. The structures for (a) $|\phi_1^{(1)}|$, (b) $|\phi_2^{(1)}|$, (c) $|\phi_3^{(1)}|$, (d) $|\phi_3^{(3)}|$, (e) $|\phi_{\mathrm{appr}}|$ with the corresponding parameters $N = k = \omega = \beta = 1$, $K = -0.442$, $\chi = 1.065$, $A = 2.13$, $\varepsilon = 0.1$.

$$
\begin{aligned}
&-440\omega^2 AK^3\chi + 32\omega^2 A\chi^3 K\Big)\cosh(\chi\zeta)^2 \\
&-192\omega^2 A\chi^3 K + 72\beta A^3 k^3\chi\omega K\Big]\sinh(\chi\zeta) \\
&-3\beta A^2 k^5\omega s_2\cosh(\chi\zeta)^4 \\
&+6\beta A^2 k^5\omega s_2\cosh(\chi\zeta)^2 = 0.
\end{aligned} \tag{46}
$$

After some elaborative calculations, the solutions of these equations yield

$$
\begin{aligned}
s_1 &= \frac{(1188K^2A - 295k^2N)}{198k^2}\mathrm{sech}(\chi\zeta) + N\mathrm{sech}(\chi\zeta)^3, \\
s_2 &= \frac{4K\chi(8\omega\chi^2 - 3\beta k^3A^2)}{\beta Ak^5}\mathrm{sech}(\chi\zeta)\tanh(\chi\zeta),
\end{aligned} \tag{47}
$$

and the parameters are fixed as

$$\lambda_2 = -\frac{\omega K(8184Nk^2K^2A + 4396N^2k^4 + 19965K^4A^2)}{33(11K^2A + 4k^2N)k^4A},$$
$$g_2 = \frac{5(1320Nk^2K^2A + 11979K^4A^2 - 292N^2k^4)}{198k^2A(11K^2A + 4k^2N)},$$
$$\beta = \frac{16\omega N}{11kA^3}, \tag{48}$$

where $A$ and $N$ are arbitrary constants. These results indicate that the scale parameters of various order can be obtained as a part of the solution so as to remove the secularities that might occur in the solution.

Therefore, an approximate solution is thus

$$\phi_{\text{appr}} = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 = \varepsilon\phi_1^{(1)} + \varepsilon^2\phi_2^{(1)} + \varepsilon^3(\phi_3^{(1)} + \phi_3^{(3)}).$$

We see that the error $E$ in the approximation thus is

$$E = \phi - \phi_{\text{appr}} = O(\varepsilon^4).$$

Figure 1 shows the structures for $\phi_1^{(1)}$, $\phi_2^{(1)}$, $\phi_3^{(1)}$, $\phi_3^{(3)}$ and $\phi_{\text{appr}}$, respectively, the more detailed parameters are directly given in the figure caption. Certainly, we can also continue calculating to get the higher-order terms. The more terms we get, the closer $\phi_{\text{appr}}$ is to $\phi$. In addition to $\phi_{\text{appr}}$, $\varepsilon$ is a smallness parameter, and $\phi_1^{(1)}$, $\phi_2^{(1)}$, $\phi_3^{(1)}$ and $\phi_3^{(3)}$ are localized functions, so the solution $\phi_{\text{appr}}$ converges.

## 5. Conclusion

In conclusion, the modified reductive perturbation method has been introduced to the symmetrical regular long wave equation by use of the stretched time and space variables in this paper. Therein, for the long wave limit, we expanded the field quantities into a power series of the wave number

$$\phi = \sum_{n=1}^{\infty} \varepsilon^n \phi_n.$$

By selecting the scaling parameters in a special way and employing the hyperbolic tangent method the higher-order terms in the perturbation expansion were obtained. Similar to the Boussinesq equation in [8], we have shown that the lowest-order term in the perturbation expansion is governed by the nonlinear Schrödinger equation, whereas the higher-order terms in the expansion are governed by the linear Schrödinger equation with an inhomogeneous term. For the small wave number region, a set of progressive wave type solutions was sought and possible secularities were removed by selecting the scaling parameters in a special way.

Furthermore, the method can be used to study not only solitary waves but also periodic waves. For instance, we can obtain the doubly periodic solution of (33) as

$$\mu = A\,\text{cn}\left[\left(\frac{Q}{2P}\right)^{1/2}\frac{A}{2m}\zeta, m\right] \tag{49}$$

with

$$\Omega = \frac{QA^2 + 2PK^2m^2 - 2QA^2m^2}{2m^2},\ P = 3\frac{\omega}{k^2},\ Q = \frac{3}{2}\beta k,$$

where $A$, $K$, $k$, $\beta$ and $\omega$ are arbitrary constants, $\text{cn}(\zeta, m)$ is the Jacobian elliptic cosine function, $m$ $(0 < m < 1)$ is modulus. When $m \to 1$, the solution (49) degenerates as hyperbolic function solution (34). Making use of the doubly periodic solution (49) and according to the same calculation steps, we will find more general expressions for $\phi_1^{(1)}$, $\phi_2^{(1)}$, $\phi_3^{(1)}$ and $\phi_3^{(3)}$, then the solution of (3) is different certainly.

We add that the method used in this paper is systematic and simple, it may be applied to study the higher-order terms in the perturbation expansion for other types of nonlinear systems.


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